Introducing motion in a circle

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Abstract

Motion in a circle troubled Newton and his contemporaries and troubles students today. This article attempts to give a clear presentation of certain aspects, particularly centripetal acceleration and centrifugal force.

A difficult concept

The founders of classical mechanics in the seventeenth century had considerable difficulty with motion in a circle. The great Dutch physicist Christiaan Huygens (1629–1695) was the first to work out the correct dynamical laws for a body moving uniformly in a circle. However, he thought that the force acting on the moving body was directed away from the centre of motion, and coined the term 'centrifugal' force to describe it [1]. Isaac Newton (1642–1727) was unclear on this point until, in 1684, assisted by Robert Hooke (1635–1702), he coined the expression 'centripetal' force and begins to speak unambiguously of a centripetal force acting on each planet and directed inwards towards the Sun [2]. Huygens' mistaken version of the concept of centrifugal force continues in use outside physics to this day, despite efforts to banish it by very distinguished physicists, such as Alexis Clairaut (1713–1765) and Heinrich Hertz (1857–1894) [3].

It may, indeed, seem counter-intuitive that a body moving uniformly in a circle should somehow be accelerating towards the centre. Richard Westfall, pre-eminent historian of Newtonian physics, describes this insight as 'the supreme act of imagination in the construction of modern dynamics' [4]. If centripetal acceleration is so difficult to grasp, can we avoid teaching it—and the non-existence of Huygens' centrifugal force—as brute facts that our students must uncomprehendingly accept, or can we build up their conceptual resources step by step so that these ideas almost seem like common sense? I believe so.

I have chosen a very introductory—almost remedial—level of presentation, intended for those A-level and foundation students who may not all be mathematically proficient. The emphasis will be on interpretation and explanation because it is here, I believe, that difficulties chiefly lie in this topic, as in so many others in physics. Of course, there are many correct ways—and unlimited numbers of unhelpful ways—of teaching motion in a circle or anything else. The structure, the pace, the use of computer animations and other visual aids, worked examples in class and coursework will, of course, depend on the background of the students. Where I have found that an aspect of this topic is generally well treated in textbooks, I have barely touched upon it here. John Warren and Tom Duncan, for example, seem to me to provide very clear explanations of weightlessness [5], so I do not deal with it. I have also benefited greatly from recent pedagogical literature on mechanics [6]. The main focus of the article is, therefore, on aspects of the interpretation of motion in a circle which I judge to be poorly treated in the literature, or entirely absent.

As every physics teacher knows, to teach the kinematics and dynamics of motion in a circle requires a knowledge on the part of the class of the geometry of the circle, angular velocity and its relationship to tangential velocity, an elementary knowledge of vector graphics [7], quite a sophisticated understanding of acceleration, and

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a knowledge of Newton's laws of motion and his law of gravity. I have encountered difficulties of understanding in each of these areas—both on my part and on the part of my students—and I shall deal one by one with those that are relevant.

The geometry of the circle

I introduce angular measure by asking students to explain each of the units found on their calculators, usually abbreviated to D R G. I ask how many 'radians' there are in a right angle, approximately, and often receive no response. I then say there are about $1\frac{1}{2}$ radians in a right angle, or about 6 in a circle, and sketch a radian. It is most important that students can imagine the radian approximately, and not simply know it as a concept associated with a formula. (Again and again, I encounter students who somehow confuse the symbol 2π with 360.) Finally, I say that the 'grade' divides the right angle into 100 units, and that it was invented during the French Revolution when all units were being decimalized [8].

What motivated the invention of the radian? This requires more careful explanation. I say that 6.283... (2π) of them are required to sweep out a complete circle. I ask why such an inconveniently large unit is used, worse: one that does not even fit into a circle a whole number of times. I then turn to the circle and show that, if the degree is used as an angular unit, the equation relating arc (s), radius (r) of a circular arc and angle (θ) has the clumsy form $s = (2\pi/360)r\theta$. If the radian [9] is used I show that this relationship becomes the far more convenient expression $s = r\theta$. It is a considerable problem for some students to know when to use degrees and when to use radians. In the theory of motion in a circle angles are always measured in radians.

I usually introduce angular velocity in terms of the orbit of the Moon around the Earth. I define it as the angle swept out per second by the imaginary line running from the Earth to the Moon[1]. I find students generally have little difficulty with the derivation of the formula $v = \omega r$ once they know $s = r\theta$. It is also important to derive for them the period formulas, $T = 2\pi r/v$ and $T = 2\pi/\omega$. The latter they find rather abstract but it can be made more intuitive by pointing out that if ω radians are swept out in one second, 1 radian will be swept out in $1/\omega$ seconds, and 2π radians will be swept out in $2\pi/\omega$ seconds, which is just T, the period of revolution.

[1] About 2.5 microradian per second.

Defining acceleration

I find that students often have a very unclear understanding of the concept of acceleration in its full generality. I do not believe it is helpful to launch it immediately with a measuring definition and even less so with a mathematical formula, nor is it particularly helpful at this point simply to say, generally, that a body is accelerating if its velocity is changing in magnitude or direction. When I ask students how acceleration is measured I usually get the reply 'metres per second squared'. When I ask them what they mean by 'second squared' the class usually falls silent. In fact, as I have attempted to show elsewhere, 'second squared' has no meaning and this notation is the result of an incoherent merging of Fourier's dimensional analysis with Gauss's quantity calculus in the late nineteenth century [10].

I have found that the most successful explanatory definition of acceleration is the 'addition of velocity to a body' since this applies to all kinds of acceleration and is rigorous. (This is the definition used by Galileo: see [11].) I also state that the direction of acceleration is the direction in which velocity is being added. The class will already be familiar with the straight line acceleration of a car. This works them into a concrete mode of thinking. I then ask how the acceleration of a car is measured. Primed by the earlier definition I may get the answer 'velocity added per second'. I then apply this to free fall under gravity and persuade them to tell me that a velocity of about 10 metres per second is added every second to a falling body. After 1 second this gives 10 metres per second, it adds up to 20 metres per second after 2 seconds, and so on. Clearly, the measuring definition of the acceleration of gravity is 10 metres per second *added every second*, or 10 metres per second, per second, for short. I discourage them from writing 10 metres per second squared and, instead, ask them to write 10 m/s/s [12]. Once this concept is clearly expressed more accurate values can be introduced.

The example of a projectile helps the class to understand that velocity can be added to a body in a direction that is perpendicular to the actual velocity of the body, or indeed in any direction whatever. This means, of course, that acceleration can be in any direction, irrespective of the present direction of velocity. A vector graph or animation showing velocities added during the motion of the projectile is essential here. At this stage I find that students appear able to grasp the directional independence of velocity and acceleration without too much difficulty.

Explaining centripetal acceleration

Consider the Moon M (figure 1(a)), travelling for a moment parallel to the tangent to its orbit. If velocity were not being added to the Moon in an inwards direction, it would continue moving off inertially along its tangent. If just the right total amount of velocity, in the right direction, is added in each short period of time, the Moon will end up after each such period having the same velocity magnitude as before, but it will be travelling along the tangent to the orbit in its new position. The angle in the diagram is greatly exaggerated to help visualize the explanation.

It may be difficult for beginners to grasp that adding a velocity can mean that the body ends up with the same velocity magnitude as before, and a velocity vector construction (known as a hodograph) helps to make this clear (figure 1(b)). The figure can be used to show that the added

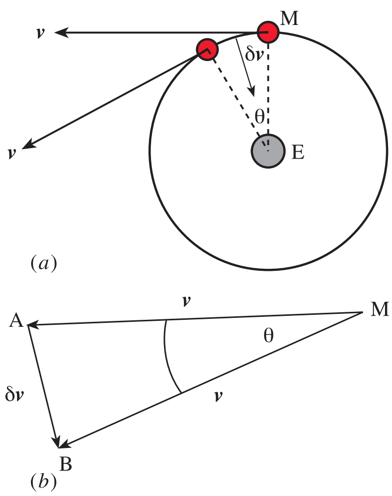

Figure 1. (a) Adding velocity to the Moon. The direction in which velocity is being added, δv, is the mean direction of acceleration. The angle θ is greatly exaggerated for convenience. (b) The hodograph (velocity vector diagram) of (a).

velocity is so directed that it reduces the velocity a little in the tangential direction, but increases it in a perpendicular direction, by just the right amounts so that the resultant velocity has the same magnitude as before—and is turned through the appropriate angle.

Since velocity is being added to the body, it is accelerating. Acceleration in a technical sense may not, therefore, mean an actual increase in speed. It is best to point out to the class here that some of them may experience a conflict with their pre-scientific understanding of acceleration, which might lead them to assume that whenever a body accelerates its speed must increase. However, all that is required scientifically for acceleration is that velocity is being added—whether or not it results in an increase, decrease or no change in speed is unimportant. In this case velocity is being added in a direction that always points towards the centre of the circle. This means that the acceleration is centripetal—towards the centre.

Bodies do not accelerate spontaneously. If a body is accelerating there must be another body or agency exerting a force upon it. For a stone whirled around by a string the body directly acting on the stone is the string, and the force is the tension of the string pulling the stone inwards. For a planet orbiting the Sun it is the gravitational field of the Sun, acting directly on the planet, that causes it to accelerate centripetally towards the Sun. The Sun does not make the planet move, but it does impress a circular form on the existing motion of the planet[2]. Without the attraction of the Sun the planet would fly off at a tangent, following its natural inertial motion.

I find it very important *not* to tell the students at this point that a centripetal acceleration is caused by a 'centripetal force' because they will surely come to believe that centripetal force is a new kind of force along with gravity, contact forces and electromagnetic forces. Furthermore, they may even think of centripetal force as something abstract, and not link in their minds to any agency which exerts it. There is, of course, no such special category of force as 'centripetal force'. I state that any force which truly causes centripetal

[2] Although the Moon and many planets have nearly circular orbits, all planetary orbits are, of course, more precisely described as elliptical. In such orbits the Sun does modify the actual speed of the planet, in a periodic manner.

acceleration in a moving body must be able to follow the latter around (as a string does), or otherwise act centripetally upon it at every point of its path. Almost any familiar dynamic agency can exert a force in such a centripetal manner. I also try to talk about this force in a concrete way, such as 'a tension acting centripetally' or 'a gravitational force acting centripetally'. Some of these points are well made in several, but not in all, textbooks [13].

Students may not yet be entirely satisfied. If the Moon is being pulled by the Earth's gravitational field, if it is continually accelerating towards the Earth, why does it not actually strike the Earth? This is an excellent question to put to the class. If they answer vaguely this is because a rather new concept is involved. If the Moon were instantaneously at rest with respect to the Earth then it would, indeed, fall towards the Earth a distance of about 17.5 km in one hour. Conversely, if there were no gravity, the tangential inertial motion of the Moon would carry it away *laterally* (not radially: see figure 2) from the Earth a distance of about 17.5 km in one hour. The combined effects of tangential inertial motion and gravity force the path of the Moon into the arc of a circle—which follows the curvature of the Earth's surface—and maintains it at the same radial distance from the Earth as it moves[3].

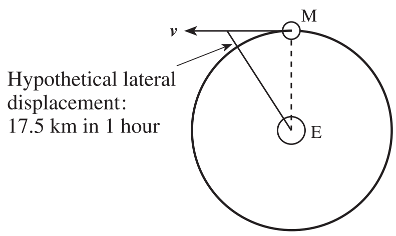

Figure 2. The hypothetical lateral displacement of the Moon from the Earth. Gravity, in the absence of lunar tangential motion, would cause the Moon to fall towards the Earth about 17.5 km in one hour. Tangential inertial motion, in the absence of gravity, would cause the Moon to move laterally away from the Earth 17.5 km in one hour. The combined effects of gravity and inertial motion force the path of the Moon into a circular arc—following the curvature of the Earth's surface—and it keeps the same radial distance from the Earth.

Centrifugal force

Teachers may groan to see yet another discussion of this troubled subject, but I believe it is a touchstone of conceptual difficulty in mechanics. I have identified at least three interpretations of centrifugal force in the literature: a valid meaning in physics, an entirely different but equally valid meaning in engineering, and a cluster of false meanings. I will consider some of the false meanings first. One writer on a respectable website states that 'When [centrifugal force] does exist, it is due to the acceleration of the mass of an object' [15]. In reality, of course, force causes acceleration (with mass as the control) and not the other way around. Here, perhaps, we have a confused application of D'Alembert's principle, which states that the negative of mass times acceleration is equal to the fictitious force required to bring the system to equilibrium [16]. D'Alembert's principle can be very perplexing, and is no longer found in undergraduate textbooks.

A recent engineering mathematics textbook states that 'The centripetal force. . . [and]. . . the centrifugal force. . . are in equilibrium at each instant of the motion' [17]. We might be forgiven for thinking that this is what theologians call the invincible blindness that can only be rectified by prayer. Unfortunately, this view is widespread. For example, many students are likely to have absorbed uncritically the statement that the Earth's attraction on the Moon is balanced by a centrifugal force. The standard physics response to this is to point out that if the force of gravity on the Moon were balanced, then according to Newton's second law there would be no lunar acceleration, since there would be no resultant force, and the Moon would fly off at a tangent. So there must be an unbalanced centripetal force acting on the Moon to maintain the circular form of its orbit. This explanation tacitly implies an inertial perspective, but students appear to adopt that naturally. Indeed, it can confuse them here to talk about inertial frameworks.

I add the following point. If there is a centrifugal force on the Moon then it must be caused by some body acting outwards on the Moon, since all forces are caused by bodies: forces are not abstract entities. Of course, no body or other agency can be identified to cause the claimed

[3] In fact, since Newton, it has been often assumed in physics that gravitational acceleration is a rapid series of sudden, small, discrete additions of velocity. Between each of these sudden additions the Moon will continue its tangential inertial motion [14].

lunar 'centrifugal force' (either in an inertial or in a revolving framework) [18].

Well, 'what about the centrifuge', some may object? I point out that the heavier particles in the centrifuge fly off to the sides more or less tangentially, rather than centrifugally, because the differential liquid pressure cannot provide enough centripetal force to maintain them in a circular path in the centrifuge. The motion of these particles is largely a tangential motion, therefore, and not a radial outwards motion, and there is no 'centrifugal force' pushing them out.

There is, however, a valid concept of centrifugal force in physics. If the observer in a frame of reference rotating with the Earth pretends for mathematical convenience that it is an inertial frame, then it becomes necessary to postulate a fictitious outwards force on a geostationary satellite to explain why it does not plunge to Earth. This is the centrifugal force of physics, an entirely fictional force [19]. It has now virtually disappeared from school and undergraduate physics textbooks because it can be highly confusing. Indeed, it is not uncommon even for physics authors to confuse the language of inertial and rotating frameworks.

But we must leave the final word to the engineers. The stresses that develop in rapidly rotating turbine blades are thought of by mechanical engineers as being due to centrifugal forces [20]. To take a simple example, an object whirled on an elastic string pulls the string outwards, creating the tension in the string. Both the inertial centrifugal force acting on the string and the elastic centripetal force acting on the moving body are reaction forces—they call each other into existence. Centrifugal and centripetal force are equal and opposite here but do not balance because they act on different bodies (figure 3).

In a rotating turbine, for example, each outer section of the blade exerts an outwards pull on the portion between it and the shaft, while at the same time the latter exerts an elastic inwards pull on the former. It is the stresses in the blades and their causes that mainly interest engineers, rather than the centripetal forces. It follows that both elastic centripetal forces and inertial centrifugal forces act in a rotating solid body [21]. What makes centrifugal force rather difficult to grasp here is that it is an inertial reaction, and such forces are not easy to visualize.

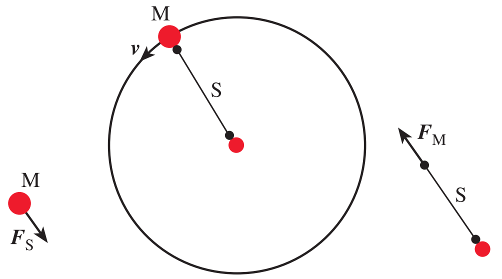

Figure 3. Centripetal force on a body M, and centrifugal force on a string S, in a gravity-free environment. $\boldsymbol{F}_S$ is the elastic centripetal force exerted by the string S on the mass M, and $\boldsymbol{F}_M$ is the centrifugal inertial reaction of the mass M on the string S. Although equal and opposite, these forces do not balance, because they act on different bodies.

Interestingly, the above considerations do not apply to many situations that concern physicists. When the gravitational field of the Earth applies a centripetal force to the Moon, for example, the Moon does not react centrifugally on the field. There is no direct reaction on a field, whether it is gravitational, electric or magnetic [22]. The Moon does, of course, act gravitationally on the Earth. True centrifugal force exists only as a reaction to macroscopic contact or binding forces.

A physics teacher might agree with this but disagree that there should be any encouragement whatever for centrifugal force in a physics class. It is difficult enough to put across the concept of centripetal force, and banish the misuse of the concept of centrifugal force. Furthermore, the centrifugal force of physics is confusing, and the centrifugal force of engineering applies only in special situations. These arguments would only allow 'bad' centrifugal force in again by the back door, and result in total incomprehension.

I agree fully that the fictional centrifugal force of physics should not even be mentioned. However, if the bulk of the class intend to go into engineering surely it would help them to have the engineering concept of centrifugal force explained clearly in their physics class beforehand. Indeed, there does seem to be some kind of unhelpful conflict, with respect to this topic, between physicists and engineers. It must be admitted, nevertheless, that this subject is subtle and the least confusing strategy for most physics groups may be to teach them centripetal force only, and leave centrifugal force to the engineers.

Calculating the centripetal acceleration

I will conclude with the derivation of an expression for centripetal acceleration. My goal here is to provide the simplest and most intuitive proof. The following version is standard, but not perhaps the nuances, so I will give it in full.

In figure 1(b) $\delta \boldsymbol{v}$ is the small velocity added during the short time δt. The mean acceleration in magnitude, therefore, is given by

$$a = \delta v/\delta t$$

where δv is the scalar magnitude of the vector $\delta \boldsymbol{v}$. A scalar derivation is easier for beginners. (Later, this can be generalized to $\boldsymbol{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$.) The direction of acceleration is, of course, that of $\delta \boldsymbol{v}$, the added velocity.

Since all three velocity scales are equal on the diagram, the numbers measuring velocity in each case can equally well be interpreted as numbers measuring the lengths of the sides of the isosceles triangle MAB. For the second part of the proof, therefore, we will think of the numbers representing v, v and δv simply as *ad hoc* numerical measures of the lengths of the sides. This sudden transition to a purely geometrical interpretation here of δv and the two v's needs to be pointed out, otherwise the explanatory jump may perplex many students.

If AB is very small it can be regarded to a good approximation as representing the magnitude of the circular arc AB linking the radii MA and MB. It follows that

$$AB = MA\,\delta\theta$$

or, in terms of their *ad hoc* measures,

$$\delta v = v\delta\theta.$$

$\delta\theta$ in figure 1(b) should be measured in radians. Now we switch back to a kinematic interpretation of δv and v. From its measuring definition, the acceleration

$$a = \delta v/\delta t = v\delta\theta/\delta t = v\omega.$$

This is perfectly accurate only when $\delta\theta$ is idealized to an infinitesimal size[4]. It does not really make sense to say that this equation 'is' the centripetal acceleration. Like all equations in physics it represents a relationship between distinct physical quantities, for example between lunar centripetal acceleration and the quantities upon which it depends—tangential velocity and angular velocity.

Since $v = \omega r$, there are two other versions of this equation,

$$a = v^2/r \quad \text{and} \quad a = \omega^2 r.$$

The version to choose depends, of course, on the problem in hand, and it should be selected to simplify the calculations as much as possible.

[4] Infinitesimals are still commonly used in physics as interim idealizations or useful fictions which immediately lead on to the formal sense of derivatives [23].

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